Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 239-246

On the intersection of geometrical structures

Ignacio Sánchez Rodríguez[1]

[1] *Departmento de Geometría y Topología*
Universidad de Granada

Abstract

We give a theorem about intersection of reductions of a principal fiber bundle. As an application, we show that the intersections of conformal and volume structures, considered as G–structures of first order, are precisely the (semi)Riemannian structures. Also, we can apply it to the intersection of both, a projective structure and the first prolongation of a volume structure, considered as G–structures of second order. A possible application for a better understanding of the General Relativity theory is pointed out.

1 Introduction

The most of differential geometrical structures commonly used can be understood as G–structures of first or second order. A G–structure of first order (or, simply, a G–structure) on a manifold M is a reduced bundle of the linear frame bundle LM with structure group a subgroup G of $GL(n,\mathbb{R})$. Examples of G–structures that we are interested in are (semi)Riemannian, conformal and volume structures. A G–structure of second order is a reduced bundle of the second order frame bundle $F^2(M)$ with structure group a subgroup G of $G^2(n)$. Examples of it are symmetric linear connections, projective structures and the first prolongations of G–structures of first order which admit symmetric connections.

In this communication we state a result about intersection of reductions of principal fiber bundles, which has an immediate lecture in terms of G–structures of first or second order. Two applications of this result are given. The first shows that (semi)Riemannian structures belonging to a given conformal structure on a manifold are in bijective correspondence with the volume structures on the manifold (for us, volume structure refers to a little generalization of volume element, which does not need the orientability of the manifold to be defined). A second application shows that a given volume structure selects a symmetric linear connection belonging to a given projective structure.

The General Relativity theory maintains that the space–time geometry is given by a Lorentzian metric structure. It is well understood ([3]) that the physical phenomenon of light propagation determines a Lorentzian conformal structure. Then, the first application suggests us to investigate in the physical motivation that would conduce to the introduction of a volume structure as an ingredient of the space-time geometry.

2 A theorem on intersection of reduced bundles

We will understand a manifold M as a C^∞, second countable, manifold of dimension n. Let G be a Lie group. Let H be a closed subgroup of G. Let $\overset{\mathrm{G,H}}{\mu} : \mathrm{G} \times (\mathrm{G/H}) \to \mathrm{G/H}$, $\overset{\mathrm{G,H}}{\mu}(a, b\mathrm{H}) \equiv \overset{\mathrm{G,H}}{\mu}_a(b\mathrm{H}) := ab\mathrm{H}$, be the natural left action of G on the homogeneous manifold G/H.

It is well known ([7, Ch.I, Prop.5.6]) the bijective correspondence between the H–reductions of a principal bundle, $P(M, \mathrm{G})$, and the sections of its associated bundle which corresponds to the left action $\overset{\mathrm{G,H}}{\mu}$. We already know that the sections of an associated bundle are in bijective correspondence with the equivariant functions of the principal bundle into the typical fibre of the associated bundle, G/H in our case. Then we can prove the following result. This result can be obtained as a consecuence of the work of Bernard ([2, Sec. I.6]) but we prefer this approach technically more clear and perfectly adapted to the applications which we are interested in.

Theorem 2.1 *Let* H, K *be two closed subgroups of a Lie group* G *such that* $\mathrm{G} = \mathrm{HK}$ *(i.e.* $\forall a \in \mathrm{G},\ \exists b \in \mathrm{H},\ c \in \mathrm{K}\colon a = bc$*). Let* $Q(M, \mathrm{H})$ *and* $R(M, \mathrm{K})$ *be two reductions of a principal bundle* $P(M, \mathrm{G})$*. Then,* $Q \cap R$ *is a reduced bundle of* P*, with* $\mathrm{H} \cap \mathrm{K}$ *as structure group.*

We give a previous lemma.

Lemma 2.2 *Let* H, K *be two closed subgroups of a Lie group* G *such that* $\mathrm{G} = \mathrm{HK}$. *Then, the application* $\rho\colon \mathrm{G}/\mathrm{K} \to \mathrm{H}/(\mathrm{H}\cap\mathrm{K})$, $\rho(a\mathrm{K}) := b(\mathrm{H}\cap\mathrm{K})$, *with* $b^{-1}a \in \mathrm{K}$, *is a diffeomorphism.*

Proof. We prove that ρ is well defined: (i) Since $\mathrm{G} = \mathrm{HK}$, given $a \in \mathrm{G}$, there exists $b \in \mathrm{H}$, with $b^{-1}a \in \mathrm{K}$, and, if other $\hat{b} \in \mathrm{H}$ also verifies $\hat{b}^{-1}a \in \mathrm{K}$ then, $\mathrm{H} \ni b^{-1}\hat{b} = (b^{-1}a)(\hat{b}^{-1}a)^{-1} \in \mathrm{K}$, which implies that $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$. ($ii$) If $a\mathrm{K} = \hat{a}\mathrm{K}$ and $b^{-1}a,\ \hat{b}^{-1}\hat{a} \in \mathrm{K}$, with $b,\ \hat{b} \in \mathrm{H}$, then $\mathrm{H} \ni b^{-1}\hat{b} = (b^{-1}a)(a^{-1}\hat{a})(\hat{b}^{-1}\hat{a})^{-1} \in \mathrm{K}$, which implies that $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$.

The application ρ is bijective: (i) It is clearly onto. (ii) If $a, \hat{a} \in \mathrm{G}$ and $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$, with $b,\ \hat{b} \in \mathrm{H}$ and $b^{-1}a,\ \hat{b}^{-1}\hat{a} \in \mathrm{K}$, then $a^{-1}\hat{a} = (b^{-1}a)^{-1}(b^{-1}\hat{b})(\hat{b}^{-1}\hat{a}) \in \mathrm{K}$, which implies that $a\mathrm{K} = \hat{a}\mathrm{K}$.

The application ρ^{-1} maps $b(\mathrm{H}\cap\mathrm{K})$ into $b\mathrm{K}$. This is an immersion ([5, Ch.II, Prop.4.4(a)]). But a bijective immersion is a diffeomorphism ([9, Ch.I, Exer.6]). Thus ρ is a diffeomorphism.

□

Proof of the theorem. Let $f\colon P \to \mathrm{G}/\mathrm{K}$ be the function $\mu^{\mathrm{G,K}}$–equivariant corresponding to $R(M,\mathrm{K})$. This means that, $\forall a \in \mathrm{G}$, $f \circ \mathcal{R}^{P}_{a} = \mu^{\mathrm{G,K}}_{a^{-1}} \circ f$, with $\mathcal{R}^{P}$ being the principal right action of G on P, and $f^{-1}(\{\mathrm{K}\}) = R$. We will prove that $Q \cap R$ is a reduction of Q, which corresponds to the equivariant function $\rho \circ f|_{Q} : Q \to \mathrm{H}/(\mathrm{H}\cap\mathrm{K})$:

(i) Let $d \in \mathrm{H}$ and $a \in \mathrm{G}$ be given, and let $b \in \mathrm{H}$ be such that $b^{-1}a \in \mathrm{K}$. We obtain that

$$\begin{aligned}(\rho \circ \mu^{\mathrm{G,K}}_{d^{-1}})(a\mathrm{K}) &= \rho(d^{-1}a\mathrm{K}) = \rho(d^{-1}b\mathrm{K}) = d^{-1}b(\mathrm{H}\cap\mathrm{K}) \\ &= d^{-1}\rho(a\mathrm{K}) = (\mu^{\mathrm{H,H\cap K}}_{d^{-1}} \circ \rho)(a\mathrm{K})\,.\end{aligned}$$

Now, given $q \in Q$, $d \in \mathrm{H}$, we obtain that

$$\begin{aligned}(\rho \circ f|_{Q} \circ \mathcal{R}^{Q}_{d})(q) &= (\rho \circ f \circ \mathcal{R}^{P}_{d})(q) = (\rho \circ \mu^{\mathrm{G,K}}_{d^{-1}} \circ f)(q) = \\ &= (\mu^{\mathrm{H,H\cap K}}_{d^{-1}} \circ \rho \circ f|_{Q})(q)\,.\end{aligned}$$

Thus the function $\rho \circ f|_{Q}$ is $\mu^{\mathrm{H,H\cap K}}$–equivariant.

(ii) Given $q \in Q$, if $(\rho\circ f|_{Q})(q) = \mathrm{H}\cap\mathrm{K}$, then we have $f(q) = f|_{Q}(q) = \rho^{-1}(\mathrm{H}\cap\mathrm{K}) = \mathrm{K}$, which implies that $q \in R$. Thus $(\rho \circ f|_{Q})^{-1}(\{\mathrm{H}\cap\mathrm{K}\}) = Q \cap R$.

The theorem follows from the fact that a reduction, $Q \cap R$, of a reduction Q of P is a reduction of P.

□

As an $(\mathrm{H} \cap \mathrm{K})$–reduction of P trivially extends to an H–reduction and to a K–reduction of P, it is inmediate to prove the following result.

Corollary 2.3 *Let* H, K *be two closed subgroups of a Lie group* G *such that* $\mathrm{G} = \mathrm{HK}$. *Let* $P(M, \mathrm{G})$ *be a principal bundle. The* $(\mathrm{H} \cap \mathrm{K})$*–reductions of* P *are precisely the intersections of* H*–reductions with* K*–reductions of* P.

3 Conformal and volume structures

Let G be a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. A G*–structure of first order* on a manifold M is a G–reduction of the linear frame bundle LM.

Let η be the standard scalar product on $\mathbb{R}^n$ of a fixed signature. We define the adjoint with respect to η, $a^\dagger$, of $a \in \mathrm{GL}(n, \mathbb{R})$ as the unique matrix such that $\eta(v, a^\dagger w) = \eta(av, w)$, $\forall v, w \in \mathbb{R}^n$. A *conformal structure* on M is a G–structure with $\mathrm{G} = \mathrm{CO}(n) := \{a \in \mathrm{GL}(n, \mathbb{R}) \colon a^\dagger a = kI,\ k > 0\}$, where I is the identity matrix in $\mathrm{GL}(n, \mathbb{R})$.

We define a *volume structure* on M as a G–structure with $\mathrm{G} = \mathrm{SL}^{\pm}(n) := \{a \in \mathrm{GL}(n, \mathbb{R}) \colon |\det(a)| = 1\}$. Note that the existence of volume structures does not depend on the orientability of M as in the case of $\mathrm{SL}(n)$–structures.

Theorem 3.1 *The (semi)Riemannian structures on* M *are the intersections of conformal and volume structures on* M.

Proof. It is clear that

$$\mathrm{CO}(n) \cap \mathrm{SL}^{\pm}(n) = \mathrm{O}(n) := \{a \in \mathrm{GL}(n, \mathbb{R}) \colon a^\dagger a = I\}.$$

Then, by the results of the previous section, we only need to prove that $\mathrm{GL}(n, \mathbb{R}) = \mathrm{CO}(n)\mathrm{SL}^{\pm}(n)$. This follows from the fact that

$$a = \left(|\det(a)|^{1/n} I\right)\left(|\det(a)|^{-1/n} a\right), \ \forall a \in \mathrm{GL}(n, \mathbb{R}) .$$

□

It is usual to define a conformal structure on M as the set $[g]$ of metric tensors which are proportional by a positive factor to a given metric tensor g on M, i.e. $g' \in [g]$ if and only if $g' = \omega g$, with $\omega \colon M \to \mathbb{R}^+$. In this context, the $\mathrm{CO}(n)$–structure P corresponding to $[g]$ is composed of all the linear frames $l \in LM$, that considered as basis of the tangent space $T_m M$ in some point $m \in M$, are orthonormal for some $g' \in [g]$.

We can also understand a volume structure $Q \subset LM$ as a selection, for every point $m \in M$, of a maximal set of basis of $T_m M$ with the same unoriented volume, in the sense of linear algebra.

It is intuitively clear that if we intersect a conformal structure P or, equivalently, $[g]$ and a volume structure Q, we are selecting, at each point $m \in M$, all the linear frames of P with the same unoriented volume defined by Q. But this procedure is equivalent to select the tensor metric in $[g]$ for which these linear frames are orthonormal. This metric is unique because two tensor metrics, which are proportional and distinct in a point $m \in M$, have orthonormal basis in $T_m M$ with different volume.

4 G–structures of second order

The *second order frame bundle* $F^2(M)$ over a manifold M (see [6, Ch.4, Sec.5] and see [8] for more details) is the principal fibre bundle, whose fibre over each point $m \in M$ is the set of *second order frames* at m, i.e. the set of 2–jets

$$\{j_0^2(x^{-1}) \colon x \text{ is a chart of } M \text{ with } x(m) = 0\}.$$

Its structure group is the Lie group of 2–jets

$$\mathrm{G}^2(n) := \{j_0^2(\phi) \colon \phi \text{ is a local diffeomorphism of } \mathbb{R}^n \text{ with } \phi(0) = 0\}.$$

Let G be a subgroup of $\mathrm{G}^2(n)$. A G–*structure of second order* on M is a G–reduction of $F^2(M)$.

Let $\mathrm{S}^2(n)$ be the set of symmetric bilinear maps $t \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, considered as an additive Lie group. For each $t \in \mathrm{S}^2(n)$, we will write $t^i_{jk} \equiv u^i(t(e_j, e_k))$, with $\{e_1, \ldots, e_n\}$ and $\{u^1, \ldots, u^n\}$ being the usual basis of $\mathbb{R}^n$ and $\mathbb{R}^{n*}$, respectively. We set $\mathrm{GL}(n, \mathbb{R}) \otimes \mathrm{S}^2(n)$ for the semidirect product of Lie groups, whose product law is given by $(a, t) \cdot (a', t') := (aa', a'^{-1} t(a', a') + t')$. There is a canonical isomorphism ([8, Lem.1], see also [1, Sec.4]) between $\mathrm{G}^2(n)$ and $\mathrm{GL}(n, \mathbb{R}) \otimes \mathrm{S}^2(n)$ given by the application that maps $j_0^2(\phi)$ into $(D\phi|_0 \, , \, D\phi|_0^{-1} \, D^2\phi|_0)$. We will identify both groups.

Examples of second order G–structures are the following:

- A symmetric linear connection of M can be identified ([6, Ch.4, Prop.7.1]) as a G–structure of second order on M, with $\mathrm{G} = \mathrm{GL}(n, \mathbb{R}) \otimes \{0\}$. It is composed of the 2–jets at 0 of the inverse of all normal charts for the symmetric linear connection ([8]).

- A projective structure on M ([6, Ch.4, Prop.7.1]) can be identified with a G–structure of second order on M, with $\mathrm{G} = \mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}$, where

$$\mathfrak{p} := \{t \in \mathrm{S}^2(n) \colon t^i_{jk} = \delta^i_j p_k + \delta^i_k p_j, \text{ for some } (p_1, \ldots, p_n) \in \mathbb{R}^{n*}\}.$$

  It is composed of the union of the G–structures of second order corresponding to the projectively equivalent symmetric connections which belong to the projective structure.

- We say that a G–structure of first order P is 1–*integrable* if it admits a symmetric linear connection. Semiriemannian, conformal and volume structures are examples of 1–integrable G–structures. The first prolongation P_1 of an 1–integrable G–structure P is ([8, Ch.4, Sec.3.3]) unique and it can be identified with an H–structure of second order with $\mathrm{H} = \mathrm{G} \otimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ denote the first prolongation of the Lie algebra $\mathfrak{g}$ of G. We can see P_1 as the set of 2–jets at 0 of the inverse of all normal charts for all symmetric linear connections that can be defined in P.

With the identifications introduced above, it can be shown the following result.

Theorem 4.1 *The intersection of a projective structure on M and the first prolongation of a volume structure on M gives a symmetric linear connection of M.*

Proof. Since the first prolongation of the Lie algebra $\mathfrak{sl}(n)$ of $\mathrm{SL}^{\pm}(n)$ is $\mathfrak{sl}(n)_1 = \{t \in \mathrm{S}^2(n) \colon t^h_{hk} = 0\}$, then

$$0 = t^h_{hk} = \delta^h_h p_k + \delta^h_k p_h = (n+1)p_k, \ \forall k \in \{1, \ldots, n\},$$

thus $t = 0$. This implies that

$$(\mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}) \cap (\mathrm{SL}^{\pm}(n) \otimes \mathfrak{sl}(n)_1) = \mathrm{SL}^{\pm}(n) \otimes \{0\}.$$

Moreover, it is readily verified that

$$\mathrm{GL}(n, \mathbb{R}) \otimes \mathrm{S}^2(n) = (\mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}) \cdot (\mathrm{SL}^{\pm}(n) \otimes \mathfrak{sl}(n)_1)$$

since $(a,t) = (a,r) \cdot (I,s), \quad \forall (a,t) \in \mathrm{GL}(n,\mathbb{R}) \rtimes \mathrm{S}^2(n)$, where

$$r^i_{jk} := \frac{1}{n+1}(\delta^i_j t^h_{hk} + \delta^i_k t^h_{hj}) \text{ and } s^i_{jk} := t^i_{jk} - \frac{1}{n+1}(\delta^i_j t^h_{hk} + \delta^i_k t^h_{hj}) \,.$$

Then, by Theorem 1, the intersection of a projective structure on M and the first prolongation of a volume structure on M is a $\mathrm{SL}^{\pm}(n) \rtimes \{0\}$–structure of second order. This extends trivially to a $\mathrm{GL}(n,\mathbb{R}) \rtimes \{0\}$–structure of second order naturally included in the projective structure.

□

In other words, this result says that a volume structure on M select a connection of a projective class of linear symmetric connections.

5 Remarks on Lorentzian geometry and General Relativity

Several geometrical structures can be derived from a (semi)Riemannian metric: a conformal structure, a symmetric linear connection and a projective structure. The fact that a metric is the intersection of a conformal structure and a volume structure allows the metric to be considered derived from the conformal and volume structures. From my point of view, the understanding of a metric structure as being composed by these two pieces can be used to gain an insight into the meaning of the General Relativity theory.

The phenomenon of light propagation is described geometrically by a field of light cones which determines a Lorentzian conformal structure. It would be very interesting to identify some substantial physical phenomenon as being represented by a volume structure. In this way, two physical principles will lead up to the Lorentzian metric proposed by the Relativity theory.

Some authors ([3], [4]) have tried to give an axiomatic approach to General Relativity by deriving the metric structure from the conformal and projective structures. The projective structure would explain the mouvement of free particles. But this approach is mathematically more complicated because some extra conditions are needed to determine a Lorentzian metric, except for a constant factor.

Acknowledgments

This work has been partially supported by Junta de Andalucía P.A.I.: FQM0203.

References

[1] E. Aguirre Dabán and I. Sánchez Rodríguez, *Explicit formulas for the 3–jet lift of a matrix group. Aplications to conformal geometry*, in: A. Pereira do Vale and M.R. Pinto, eds., *Proceedings of the 1st International Meeting on Geometry and Topology* (Universidade do Minho, Braga, Portugal, 1998) 191–205.

[2] D. Bernard, *Sur la géométrie différentielle des G-structures*, Ann. Inst. Fourier, Grenoble, **10**, 151-270, (1960).

[3] J. Ehlers, F.A.E. Pirani and A. Schild, *The geometry of free fall and light propagation*, in: L. O'Raifeartaigh, ed., *General Relativity* (Oxford, 1972) 63–84.

[4] J. Ehlers, *Survey of General Relativity Theory*, in: W. Israel, ed., *Relativity, Astrophysics and Cosmology* (Dordrecht–Holland, 1973) 1–124.

[5] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[6] S. Kobayashi, *Transformation Groups in Differential Geometry* (Springer, Heidelberg, 1972).

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol.I* (John Wiley, New York, 1963).

[8] I. Sánchez Rodríguez, *Conexiones en el fibrado de referencias de segundo orden. Conexiones conformes* (Thesis, Department of Geometry and Topology, Complutense University, Madrid, 1994).

[9] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Scott, Foresman and Co., Glenview, Illinois, 1971).